Classical Mechanics
Notes for the dumb Jose Mathew joining MSc
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Contents

Chapter 1

Central Force

1.1 Central Force

A *central force* is a force that acts along the line joining a particle and a fixed point, typically the origin. The magnitude of the force depends only on the radial distance r between the particle and the center of force, but not on the direction. Mathematically, it is expressed as:

$$\mathbf{F} = F(r)\hat{\mathbf{r}},$$

where: - $\hat{\mathbf{r}}$ is the unit vector in the radial direction, - $F(r)$ is the magnitude of the force, which depends only on r.

1.1.1 Characteristics of a Central Force

- **Radial Nature**: The force acts along or opposite to $\mathbf{r}$, and its direction is either attractive ($F(r) < 0$) or repulsive ($F(r) > 0$).
- **Conservative Force**: Central forces are usually conservative, meaning they can be derived from a scalar potential $V(r)$:

$$\mathbf{F} = -\nabla V(r) = -\frac{dV}{dr}\hat{\mathbf{r}}.$$

- **Angular Momentum Conservation**: Since the force acts radially, there is no torque about the center. This ensures that angular momentum $\mathbf{L}$ is conserved:

$$\frac{d\mathbf{L}}{dt} = 0.$$

- **Motion in a Plane**: Conservation of angular momentum confines the motion to a plane perpendicular to $\mathbf{L}$, reducing the problem to two dimensions.

1.1.2 Examples of Central Forces

- **Gravitational Force**:

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}},$$

where G is the gravitational constant.

- **Electrostatic Force**:
$$\mathbf{F} = \frac{kq_1q_2}{r^2}\hat{\mathbf{r}},$$
where k is Coulomb's constant.
- **Spring Force (Radial Hooke's Law)**:
$$\mathbf{F} = -kr\hat{\mathbf{r}},$$
where k is the spring constant.

1.1.3 Applications

- **Planetary Motion**: Central forces govern planetary orbits, as described by Newton's law of gravitation and Kepler's laws.
- **Atomic Models**: The electrostatic force between electrons and nuclei in atoms is a central force.
- **Oscillatory Systems**: Radial spring forces model harmonic oscillators in various fields of physics.

In summary, central forces simplify the analysis of motion by reducing the dynamics to two dimensions and conserving angular momentum. They are fundamental to understanding natural phenomena ranging from celestial mechanics to molecular dynamics.

1.2 Derivation of $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{R}$, m_1, m_2, and $\mathbf{r}$

1.2.1 The diagram

This document provides a detailed step-by-step derivation of $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of the center of mass $\mathbf{R}$, the masses m_1 and m_2, and the relative position vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

1.2.2 Step 1: Definition of Center of Mass

The center of mass (COM) position vector $\mathbf{R}$ is defined as:
$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$
This represents the weighted average of the positions of the two masses.

1.2.3 Step 2: Define Relative Position Vector

The relative position vector $\mathbf{r}$ is defined as:
$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$
Rearranging, we can write:
$$\mathbf{r}_2 = \mathbf{r} + \mathbf{r}_1.$$
This relationship will be used to substitute $\mathbf{r}_2$ in the center of mass equation.

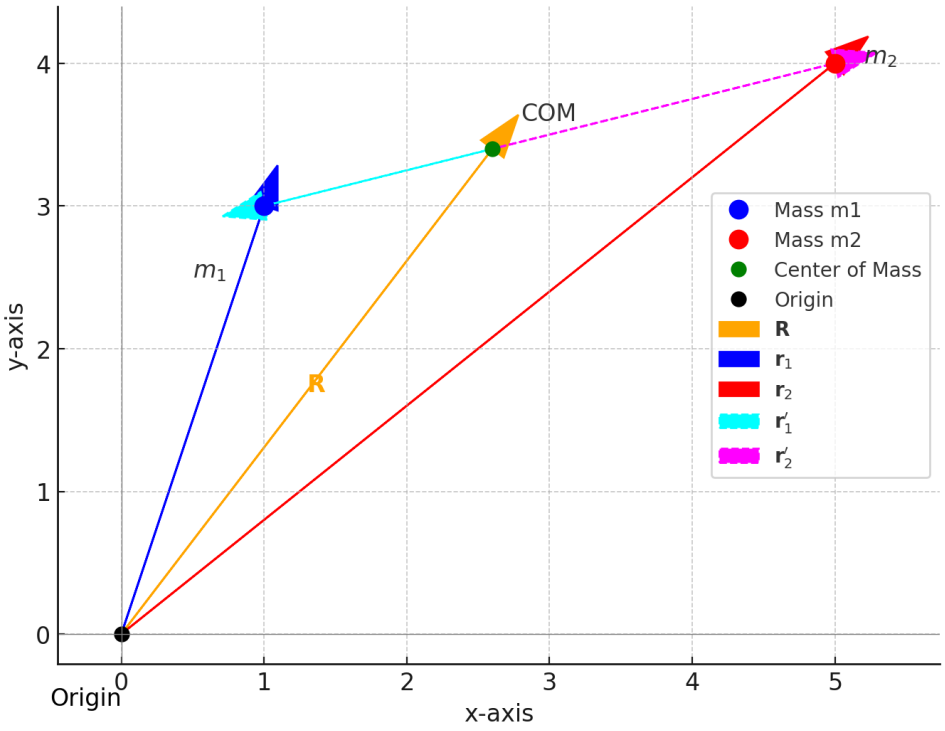

Figure 1.1: Illustration of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}$, and $\mathbf{R}$.

1.2.4 Step 3: Derive $\mathbf{r}_1$

1. Substitute $\mathbf{r}_2 = \mathbf{r} + \mathbf{r}_1$ into the COM equation:
$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2(\mathbf{r} + \mathbf{r}_1)}{m_1 + m_2}.$$
2. Expand the numerator:
$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r} + m_2\mathbf{r}_1}{m_1 + m_2}.$$
3. Combine terms involving $\mathbf{r}_1$:
$$\mathbf{R} = \frac{(m_1 + m_2)\mathbf{r}_1 + m_2\mathbf{r}}{m_1 + m_2}.$$
4. Eliminate the denominator:
$$(m_1 + m_2)\mathbf{R} = (m_1 + m_2)\mathbf{r}_1 + m_2\mathbf{r}.$$
5. Isolate $\mathbf{r}_1$:
$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}.$$

1.2.5 Step 4: Derive $\mathbf{r}_2$

1. Substitute $\mathbf{r}_1 = \mathbf{r}_2 - \mathbf{r}$ into the COM equation:
$$\mathbf{R} = \frac{m_1(\mathbf{r}_2 - \mathbf{r}) + m_2\mathbf{r}_2}{m_1 + m_2}.$$

2. Expand the numerator:
$$\mathbf{R} = \frac{m_1\mathbf{r}_2 - m_1\mathbf{r} + m_2\mathbf{r}_2}{m_1 + m_2}.$$

3. Combine terms involving $\mathbf{r}_2$:
$$\mathbf{R} = \frac{(m_1 + m_2)\mathbf{r}_2 - m_1\mathbf{r}}{m_1 + m_2}.$$

4. Eliminate the denominator:
$$(m_1 + m_2)\mathbf{R} = (m_1 + m_2)\mathbf{r}_2 - m_1\mathbf{r}.$$

5. Isolate $\mathbf{r}_2$:
$$\mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

1.2.6 Final Results

1. Position of $\mathbf{r}_1$:
$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}.$$
2. Position of $\mathbf{r}_2$:
$$\mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

1.2.7 Symmetry Between $\mathbf{r}_1$ and $\mathbf{r}_2$

The expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$ exhibit symmetry:
$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.$$
This symmetry reflects the balance of the system about the center of mass.

1.3 Reduction of Two-Body Problem to One-Body Problem

This document derives the kinetic energy (KE) and potential energy (V) for a two-body system and reduces the two-body problem to an equivalent one-body problem.

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1.3.1 Step 1: Kinetic Energy and Potential Energy in Terms of $\mathbf{r}_1$ and $\mathbf{r}_2$

The total kinetic energy of the two-body system is given by:
$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2.$$

The potential energy depends only on the relative position vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$:
$$V = V(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}).$$

—

1.3.2 Step 2: Rewrite KE and V in Terms of r, R, m_1, and m_2

1. The center of mass $\mathbf{R}$ is defined as:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$

The relative position vector is:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

2. Using these, the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ can be written as:

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

3. The velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ become:

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{R}} - \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}}, \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} + \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}}.$$

4. Substitute into the kinetic energy:

$$T = \frac{1}{2}m_1\left(\dot{\mathbf{R}} - \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} + \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}}\right)^2.$$

5. Expand the squares and simplify using $m_1 + m_2$:

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2,$$

where μ is the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

6. The potential energy becomes:

$$V = V(\mathbf{r}).$$

—

1.3.3 Step 3: Lagrangian for the System

The Lagrangian L is:

$$L = T - V.$$

Substituting the expressions for T and V:

$$L = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(\mathbf{r}).$$

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1.3.4 Step 4: R as a Cyclic Coordinate

1. The center of mass $\mathbf{R}$ appears only in the kinetic energy term $\frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2$ and does not appear in the potential energy V. 2. Therefore, $\mathbf{R}$ is a cyclic coordinate, and its conjugate momentum:

$$\mathbf{P} = (m_1 + m_2)\dot{\mathbf{R}}$$

is conserved.

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1.3.5 Step 5: Reduced Lagrangian in Terms of μ and r

After separating the motion of the center of mass, the reduced Lagrangian describes the relative motion:

$$L_{\text{reduced}} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(\mathbf{r}).$$

This Lagrangian describes a single particle of mass μ moving under the influence of the potential $V(\mathbf{r})$.

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1.3.6 Conclusion: Reduction of Two-Body Problem to One-Body Problem

1. The original two-body problem, described by the positions $\mathbf{r}_1$ and $\mathbf{r}_2$, has been reduced to two independent problems: - Motion of the center of mass $\mathbf{R}$, which is uniform if no external forces act. - Relative motion of a single particle of mass μ under the potential $V(\mathbf{r})$.

2. This simplification is achieved by introducing the center of mass and relative position coordinates, effectively reducing the degrees of freedom from two bodies to one equivalent body.

1.4 Deriving the Euler-Lagrange Equation for the Reduced Lagrangian

Let's explore the process of deriving the Euler-Lagrange equation for the reduced Lagrangian in the context of a two-body system. To make it engaging, imagine we're discussing this as part of a one-on-one mentorship session, walking through the steps interactively.

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1.4.1 Step 1: Expressing the Full Lagrangian

In a two-body system, the relative position vector $\mathbf{r}$ is written as:

$$\mathbf{r} = r\hat{\mathbf{r}},$$

where: - r: radial distance, - $\hat{\mathbf{r}}$: unit vector in the radial direction.

Using spherical coordinates (r, θ, ϕ), the Lagrangian becomes:

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r),$$

where: - $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass, - $V(r)$ is the potential energy depending only on r, - $\dot{r}, \dot{\theta}, \dot{\phi}$ are time derivatives of the spherical coordinates.

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1.4.2 Step 2: Motion Constrained to a Plane

A central force problem inherently has rotational symmetry about the center of mass. Here's why the motion of the two-body system always occurs in a plane:

1.4.2.1 Step 2: Angular Momentum

In a central force problem, the motion is confined to a plane due to the conservation of angular momentum. Let us derive this step by step:

1. **Central Force Definition**:

The central force is given as:

$$\mathbf{F} = F(r)\hat{\mathbf{r}},$$

where $F(r)$ depends only on the radial distance r, and $\hat{\mathbf{r}}$ is the unit vector in the radial direction.

2. **Torque is Zero**:

The torque $\boldsymbol{\tau}$ is:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Substituting $\mathbf{F} = F(r)\hat{\mathbf{r}}$:

$$\boldsymbol{\tau} = \mathbf{r} \times (F(r)\hat{\mathbf{r}}) = 0,$$

because $\mathbf{r}$ and $\hat{\mathbf{r}}$ are collinear. Hence, there is no torque.

3. **Angular Momentum Conservation**:

The angular momentum $\mathbf{L}$ of the system is:

$$\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}.$$

Taking the time derivative:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times \mu\dot{\mathbf{r}}\right).$$

4. **Product Rule for the Derivative**:

Using the product rule:

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times \mu\dot{\mathbf{r}} + \mathbf{r} \times \mu\ddot{\mathbf{r}}.$$

- The first term is $\dot{\mathbf{r}} \times \mu\dot{\mathbf{r}} = 0$, since the cross product of a vector with itself is zero. - The second term is:

$$\mathbf{r} \times \mu\ddot{\mathbf{r}},$$

where $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{\mu} = F(r)\hat{\mathbf{r}}$. Substituting:

$$\mathbf{r} \times \mu\ddot{\mathbf{r}} = \mathbf{r} \times (F(r)\hat{\mathbf{r}}) = 0,$$

because $\mathbf{r}$ and $\hat{\mathbf{r}}$ are collinear.

5. **Conclusion**:

Since both terms are zero, the time derivative of angular momentum is zero:

$$\frac{d\mathbf{L}}{dt} = 0.$$

Hence, angular momentum $\mathbf{L}$ is conserved, and the motion is confined to a plane perpendicular to $\mathbf{L}$.

6. **Direction of Angular Momentum:** The vector $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\dot{\mathbf{r}}$. This implies the motion lies entirely within a fixed plane perpendicular to $\mathbf{L}$.

7. **Simplification:** By choosing the plane of motion to coincide with the equatorial plane $\theta = \pi/2$, we simplify the dynamics. The coordinate ϕ remains to describe angular motion within this plane.

The reduced Lagrangian now becomes:

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r).$$

Here, ϕ is a cyclic coordinate, meaning it does not explicitly appear in the Lagrangian but contributes through the conserved angular momentum.

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1.4.3 Step 3: Euler-Lagrange Equations

Radial Motion The Euler-Lagrange equation for r is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0.$$

1. **Compute $\frac{\partial L}{\partial \dot{r}}$:**
$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}.$$
2. **Compute the time derivative of $\frac{\partial L}{\partial \dot{r}}$:**
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \mu\ddot{r}.$$
3. **Compute $\frac{\partial L}{\partial r}$:** From the Lagrangian:
$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r),$$
the partial derivative with respect to r is:
$$\frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{\partial V}{\partial r}.$$
4. **Substitute into the Euler-Lagrange Equation:**
$$\mu\ddot{r} - \mu r\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0.$$

1.4.4 Angular Motion

see 1.9 The Euler-Lagrange equation for ϕ is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0.$$

Since ϕ is cyclic, $\frac{\partial L}{\partial \phi} = 0$, and:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0.$$

The conserved angular momentum is:

$$L_z = \mu r^2\dot{\phi}.$$

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1.4.5 Step 4: Effective Potential

Using the conserved angular momentum, the angular term can be absorbed into an effective potential:

$$V_{\text{eff}}(r) = V(r) + \frac{L_z^2}{2\mu r^2}.$$

The radial equation becomes:

$$\mu \ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r}.$$

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1.4.6 Centrifugal Potential

The *centrifugal potential* arises naturally in problems involving central forces when motion is analyzed in polar or spherical coordinates. It is an effective potential term that accounts for the angular momentum of the system, reflecting the "apparent force" experienced due to rotation. Mathematically, it is expressed as:

$$V_{\text{centrifugal}}(r) = \frac{L_z^2}{2\mu r^2},$$

where L_z is the conserved angular momentum, μ is the reduced mass, and r is the radial distance. This term increases sharply as r decreases, creating a barrier that prevents the particle from collapsing into the central force. The centrifugal potential, combined with the actual potential $V(r)$, forms the effective potential $V_{\text{eff}}(r)$, which governs the radial motion of the particle. This concept is essential in explaining phenomena such as stable orbits and equilibrium distances in celestial mechanics and atomic physics.

1.4.7 Conclusion

By leveraging the rotational symmetry of the two-body problem, we constrained the motion to a single plane. The azimuthal coordinate ϕ remains as a cyclic coordinate, contributing to the dynamics through the conserved angular momentum. The radial motion is fully described by an effective potential, simplifying the analysis significantly. Note, that we have retained ϕ even though ϕ is cyclic. But by pointing out R is cyclic we had ignored R from the entire calculation. Next section explains why.

1.5 Revisiting ignorable coordinate and cyclic coordinate in Central force problems

1.5.1 Angular Momentum: Cyclic but Not Ignorable Coordinate

This document discusses the distinction between cyclic and ignorable coordinates in the context of angular momentum in central force problems, as exemplified in Kepler's problem.

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1.5.2 Angular Momentum as a Cyclic Coordinate

1. In a central force problem (e.g., gravitational or electrostatic interaction): - The Lagrangian in spherical coordinates (or polar coordinates in 2D) is given by:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r),$$

where: - r: radial distance, - ϕ: azimuthal angle, - $\dot{r}$ and $\dot{\phi}$: time derivatives of r and ϕ, respectively.

2. The azimuthal angle ϕ is a cyclic coordinate because it does not explicitly appear in the Lagrangian. Its conjugate momentum is:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi},$$

which corresponds to the angular momentum about the axis of symmetry. Since ϕ is cyclic, p_ϕ is conserved:

$$p_\phi = \text{constant}.$$

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1.5.3 Not an Ignorable Coordinate

While ϕ is cyclic, it is not ignorable because:

1. The conserved angular momentum p_ϕ influences the radial dynamics through the effective potential:

$$V_{\text{eff}}(r) = V(r) + \frac{p_\phi^2}{2mr^2}.$$

This term modifies the radial equation of motion, linking the angular motion (ϕ) to the radial motion (r).

2. Ignorable coordinates are typically redundant or do not affect the system's dynamics. In this case, ϕ is not redundant because its conjugate momentum directly contributes to the energy and dynamics of the system.

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1.5.4 Key Distinction: Cyclic vs Ignorable Coordinates

- **Cyclic Coordinate**: - A coordinate that does not explicitly appear in the Lagrangian. - Leads to a conserved quantity (e.g., angular momentum).

- **Ignorable Coordinate**: - A broader term that includes coordinates irrelevant to the dynamics or omitted due to symmetry or constraints.

- **In this problem**: - ϕ is cyclic because it is absent from the Lagrangian. - ϕ is not ignorable because its conserved momentum influences the effective potential and the system's radial dynamics.

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1.6 Summary

The azimuthal angle ϕ in a central force problem is a cyclic coordinate, leading to the conservation of angular momentum. However, it is not an ignorable coordinate because its conjugate momentum p_ϕ affects the radial motion through the effective potential.

This distinction highlights the nuanced role of cyclic coordinates in shaping the system's dynamics.

1.7 Differential Equation for r Using the Energy Method

To derive the differential equation for the radial motion r, we use the energy conservation principle.

1.7.0.1 Step 1: Write the Total Energy

The total energy E of the system is the sum of the kinetic energy T and the effective potential $V_{\text{eff}}(r)$:

$$E = T + V_{\text{eff}}(r),$$

where: - T is the kinetic energy, - $V_{\text{eff}}(r)$ is the effective potential energy.

1. Kinetic Energy:

$$T = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2,$$

where μ is the reduced mass, $\dot{r}$ is the radial velocity, and $\dot{\phi}$ is the angular velocity.

2. Effective Potential: The effective potential is:

$$V_{\text{eff}}(r) = V(r) + \frac{L_z^2}{2\mu r^2},$$

where $L_z = \mu r^2\dot{\phi}$ is the conserved angular momentum.

3. Total Energy: Substituting T and $V_{\text{eff}}(r)$ into the total energy:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + V(r) + \frac{L_z^2}{2\mu r^2}.$$

Using $\dot{\phi}^2 = \frac{L_z^2}{\mu^2 r^4}$, the total energy simplifies to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L_z^2}{2\mu r^2} + V(r).$$

Hence:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r),$$

where $V_{\text{eff}}(r) = V(r) + \frac{L_z^2}{2\mu r^2}$.

1.7.0.2 Step 2: Solve for $\dot{r}$

From the total energy equation:

$$\frac{1}{2}\mu\dot{r}^2 = E - V_{\text{eff}}(r).$$

Multiply through by $2/\mu$:

$$\dot{r}^2 = \frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right).$$

Taking the square root:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}.$$

1.7.0.3 Step 3: Differential Equation for r

The differential equation for r is:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - V_{\text{eff}}(r)\right)}.$$

This equation relates the radial position r to the total energy E, the effective potential $V_{\text{eff}}(r)$, and the reduced mass μ.

1.8 Orbits in a Central Force Field

The motion of a particle in a central force field is determined by its **total energy** (E) and the **effective potential** ($V_{\text{eff}}(r)$). For a potential $V(r) = -\frac{k}{r}$, the effective potential is:

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{L^2}{2\mu r^2},$$

where:

- $k > 0$: strength of the attractive potential,
- L: angular momentum,
- μ: reduced mass.

1.8.1 Possible Scenarios

1.8.1.1 1. $E < 0$: Bound Orbit

- **Nature:** The particle's total energy is less than zero, meaning it is trapped in the potential well.
- **Radial Motion:** The particle oscillates between two turning points ($r_{\min}$ and $r_{\max}$) where $E = V_{\text{eff}}(r)$. At these points, the radial velocity $\dot{r} = 0$, and the particle reverses direction.
- **Trajectory:** The orbit is **elliptical** (or circular if $E = V_{\text{eff}}(r_{\min})$). Conservation of angular momentum confines the motion to a plane.

1.8.1.2 2. $E = 0$: Parabolic Trajectory

- **Nature:** The particle has just enough energy to escape the potential, moving along a parabolic trajectory.
- **Radial Motion:** There is a single turning point ($r_{\min}$) where $E = V_{\text{eff}}(r)$. Beyond $r_{\min}$, the particle moves outward indefinitely as $r \to \infty$.
- **Forbidden Region:** If $r < r_{\min}$, $\dot{r}^2 < 0$, which is non-physical, indicating that the particle cannot approach closer than $r_{\min}$.
- **Trajectory:** The orbit is **parabolic**, representing the escape of the particle at precisely the escape velocity.

1.8.1.3 3. $E > 0$: Unbound Orbit

- **Nature:** The particle's energy exceeds the maximum of $V_{\text{eff}}(r)$, allowing it to escape the central force field.
- **Radial Motion:** There is one turning point ($r_{\min}$) where $E = V_{\text{eff}}(r)$. The particle moves outward indefinitely for $r > r_{\min}$, and $r \to \infty$ as $t \to \infty$.
- **Forbidden Region:** For $r < r_{\min}$, $\dot{r}^2 < 0$, making this region inaccessible.
- **Trajectory:** The orbit is **hyperbolic**, indicating an unbound trajectory.

1.8.1.4 4. $E = V_{\text{eff}}(r)$: Circular Orbit (Special Case)

- **Nature:** At this energy, the particle is momentarily at rest in the radial direction ($\dot{r} = 0$).
- **Radial Behavior:**
  - If $E = V_{\text{eff}}(r_{\min})$, the particle remains at the same radius r, resulting in a **circular orbit**.
  - For any other r, this represents a turning point of the radial motion, where the particle reverses direction.

1.8.2 Summary of Orbit Types

Energy (E)	Orbit Type	Key Characteristics
$E < 0$	Bound (Elliptical)	Constrained between $r_{\min}$ and $r_{\max}$.
$E = 0$	Parabolic (Escape)	Single turning point $r_{\min}$, particle escapes.
$E > 0$	Unbound (Hyperbolic)	Single turning point $r_{\min}$, particle escapes infinitely.
$E = V_{\text{eff}}(r)$	Circular Orbit (Special Case)	Particle remains at a fixed radius r.

1.8.3 Graphical Representation

Below is the graph showing the effective potential $V_{\text{eff}}(r)$, the physical potential $V_{\text{physical}}(r)$, and the centrifugal potential $V_{\text{centrifugal}}(r)$. Turning points and orbit types are determined based on the energy levels E relative to $V_{\text{eff}}(r)$.

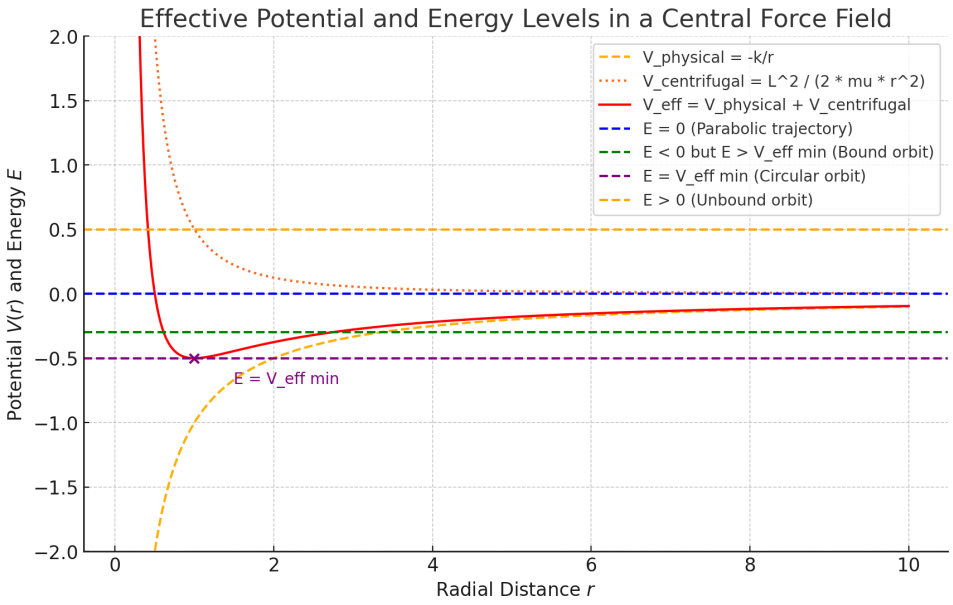

Figure 1.2: Effective Potential and Energy Levels in a Central Force Field

1.9 Angular Momentum as a First Integral

Definition of a First Integral:
A first integral is a conserved quantity derived directly from the equations of motion of a system. It provides a constraint or relationship that holds throughout the motion. A first integral corresponds to a physical quantity that remains constant due to a symmetry in the system.

1.9.1 Angular Momentum as a First Integral

Angular momentum $\mathbf{L}$ is the *first integral of the equations of motion* in systems with rotational symmetry (central forces). Let us derive this step by step.

1. Central Force and Torque:
For a central force $\mathbf{F}$ that depends only on the radial distance r:

$$\mathbf{F} = F(r)\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction.

The torque $\boldsymbol{\tau}$ about the origin is:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Since $\mathbf{r}$ and $\mathbf{F}$ are collinear, their cross product is zero:

$$\boldsymbol{\tau} = \mathbf{r} \times (F(r)\hat{\mathbf{r}}) = 0.$$

No net torque implies that the angular momentum $\mathbf{L}$ is conserved:

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = 0.$$

2. Angular Momentum Equation:
The angular momentum $\mathbf{L}$ is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{p} = m\dot{\mathbf{r}}$ is the linear momentum of the particle. Taking the time derivative:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}).$$

Using the product rule:

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}}.$$

- The first term, $\dot{\mathbf{r}} \times m\dot{\mathbf{r}}$, is zero because the cross product of a vector with itself is zero. - The second term is:

$$\mathbf{r} \times m\ddot{\mathbf{r}},$$

where $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} = F(r)\hat{\mathbf{r}}$. Substituting:

$$\mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{r} \times (F(r)\hat{\mathbf{r}}) = 0,$$

because $\mathbf{r}$ and $\hat{\mathbf{r}}$ are collinear.

Thus:

$$\frac{d\mathbf{L}}{dt} = 0,$$

which confirms that angular momentum is conserved.

3. Angular Momentum as a First Integral of Motion:
The conservation of angular momentum $\mathbf{L}$ provides a direct relationship between the motion and the symmetry of the system: - *First Integral of Rotational Motion:* Angular momentum is the integral of the rotational equations of motion. It reflects the fact that no external torque acts on the system. - *Connection to Symmetry:* According to Noether's theorem, every continuous symmetry corresponds to a conserved quantity. For rotational symmetry about the origin, the conserved quantity is angular momentum.

1.9.1.1 Implications of Angular Momentum Conservation

- Angular momentum is a constraint that reduces the degrees of freedom in the system. - In planar motion (e.g., central force problems), the magnitude of $\mathbf{L}$ is:

$$L = |\mathbf{L}| = mr^2\dot{\phi},$$

where $\dot{\phi}$ is the angular velocity. This relationship simplifies the equations of motion by eliminating the explicit dependence on ϕ.

1.9.2 Conclusion

Angular momentum is a first integral of the equations of motion in systems with central forces. It reflects the rotational invariance of the system and simplifies the dynamics by reducing the number of variables needed to describe the motion.

1.10 Relationship Between Euler-Lagrange Equation and the Hamiltonian. Total Energy as a first integral

The connection between the Euler-Lagrange (EL) equation and the Hamiltonian arises from the conservation laws and the structure of the Lagrangian. Let's explore this step by step.

1.10.1 1. Euler-Lagrange Equation Recap

The Euler-Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

where: - $L = T - V$ is the Lagrangian, - q_i are the generalized coordinates, - $\dot{q}_i$ are their time derivatives.

This equation describes how a system evolves dynamically under the influence of forces.

1.10.2 2. Integrating the EL Equation

The EL equation expresses the balance of forces in terms of generalized coordinates. However: - **Direct integration of the EL equation** (with respect to t) gives the generalized momentum conjugate to q_i:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

- This momentum is conserved if $\frac{\partial L}{\partial q_i} = 0$, i.e., if q_i is a cyclic coordinate.

1.10.3 3. Hamiltonian Connection

The Hamiltonian H is not directly obtained by integrating the EL equation but is defined as:

$$H = \sum_i \dot{q}_i p_i - L,$$

where: - $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the generalized momentum.

If the Lagrangian L does not explicitly depend on time t, the Hamiltonian H becomes a conserved quantity (the total energy of the system).

1.10.4 4. Energy Conservation from EL Equations

The Hamiltonian emerges naturally from the EL equations if the system is time-invariant: 1. Start with the EL equation for q_i:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}.$$

2. Multiply through by $\dot{q}_i$ and sum over all i:

$$\sum_i \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_i \dot{q}_i \frac{\partial L}{\partial q_i}.$$

3. Using the total time derivative of the Lagrangian:

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right),$$

and the EL equations, it follows that:

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0.$$

Thus, the Hamiltonian $H = \sum_i \dot{q}_i p_i - L$ is conserved.

1.10.5 4. Energy Conservation from EL Equations, With clear elaborate steps

The Hamiltonian arises naturally from the Euler-Lagrange equations if the system is time-invariant. Let us derive this step by step.

Step 1: Start with the Euler-Lagrange Equation
The Euler-Lagrange (EL) equation for a generalized coordinate q_i is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

Here: - $\frac{\partial L}{\partial \dot{q}_i}$ is the generalized momentum p_i, i.e., $p_i = \frac{\partial L}{\partial \dot{q}_i}$, - $\frac{\partial L}{\partial q_i}$ represents the force term derived from the potential.

The EL equation ensures that the forces balance the change in momentum for each coordinate.

Step 2: Multiply by $\dot{q}_i$ and Sum Over All Coordinates
To derive energy conservation, multiply the EL equation for q_i by the velocity $\dot{q}_i$, and sum over all i:

$$\sum_i \dot{q}_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) = 0.$$

This expands to:

$$\sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} = 0.$$

Step 3: Analyze the First Term
The first term is:

$$\sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}.$$

Using the product rule for differentiation:

$$\dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}.$$

Summing over all i, this becomes:

$$\sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i.$$

Step 4: Analyze the Second Term
The second term in the EL equation expansion is:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial q_i}.$$

By the definition of the total time derivative of L, we know:

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right).$$

Rearranging for $\sum_i \dot{q}_i \frac{\partial L}{\partial q_i}$:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial q_i} = \frac{dL}{dt} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i.$$

Step 5: Substitute Back into the Expanded EL Equation
Substituting both terms into the expanded EL equation:

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \frac{dL}{dt} = 0.$$

Reorganizing:

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0.$$

This implies that the quantity inside the parentheses is conserved:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{constant}.$$

Step 6: Interpret the Result as the Hamiltonian
The conserved quantity is the Hamiltonian H, defined as:

$$H = \sum_i \dot{q}_i p_i - L,$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the generalized momentum.

If the Lagrangian L is time-invariant (does not explicitly depend on t), the Hamiltonian H represents the total energy of the system:

$$H = T + V = E.$$

Conclusion:
By multiplying the Euler-Lagrange equations with $\dot{q}_i$ and summing over all coordinates, we derive the conservation of the Hamiltonian. This conserved quantity reflects the total energy of the system when the Lagrangian is time-independent.

1.10.6 Conclusion

While integrating the EL equation does not directly give the Hamiltonian H, it reveals conserved momenta and forms the basis for deriving the Hamiltonian. The Hamiltonian emerges as a conserved quantity when the Lagrangian is time-invariant, and it can be constructed from the system's Lagrangian using its definition.

1.11 Hamiltonian is Total Energy if L is independent of time

Why, when L is independent of time, Hamiltonian is Total Energy, and hence leading us to conclude that TE is a first integral of motion which is conserved

1.11.0.1 Step 1: Hamiltonian Definition

The Hamiltonian H is defined as:

$$H = \sum_i \dot{q}_i p_i - L,$$

where:

- $p_i = \frac{\partial L}{\partial \dot{q}_i}$: Generalized momentum,
- $L = T - V$: Lagrangian,
- T: Kinetic energy,
- V: Potential energy.

1.11.0.2 Step 2: Kinetic Energy in Terms of Velocities

The kinetic energy T is quadratic in generalized velocities:

$$T = \frac{1}{2} \sum_{i,j} a_{ij}(q) \dot{q}_i \dot{q}_j,$$

where:

- $a_{ij}(q)$: Coefficients dependent on the generalized coordinates q_i,
- $a_{ij} = a_{ji}$: Symmetry of the kinetic energy terms.

1.11.0.3 Step 3: Generalized Momentum

The generalized momentum p_i is defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Since $L = T - V$ and V does not depend on velocities:

$$p_i = \frac{\partial T}{\partial \dot{q}_i}.$$

Substituting $T = \frac{1}{2}\sum_{j,k} a_{jk}(q)\dot{q}_j\dot{q}_k$:

$$p_i = \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\sum_{j,k} a_{jk}(q)\dot{q}_j\dot{q}_k\right).$$

Differentiating:

$$\frac{\partial}{\partial \dot{q}_i}\left(a_{jk}\dot{q}_j\dot{q}_k\right) = \delta_{ij}a_{jk}\dot{q}_k + \delta_{ik}a_{jk}\dot{q}_j.$$

Using the symmetry $a_{ij} = a_{ji}$, this simplifies to:

$$p_i = \sum_j a_{ij}(q)\dot{q}_j.$$

1.11.0.4 Step 4: Substitute $\sum_i \dot{q}_i p_i$

Now calculate $\sum_i \dot{q}_i p_i$:

$$\sum_i \dot{q}_i p_i = \sum_i \dot{q}_i \left(\sum_j a_{ij}(q)\dot{q}_j\right).$$

Rearranging the summations:

$$\sum_i \dot{q}_i p_i = \sum_{i,j} a_{ij}(q)\dot{q}_i\dot{q}_j.$$

This is twice the kinetic energy T, since:

$$T = \frac{1}{2}\sum_{i,j} a_{ij}(q)\dot{q}_i\dot{q}_j.$$

Thus:

$$\sum_i \dot{q}_i p_i = 2T.$$

1.11.0.5 Step 5: Hamiltonian Expression

Substitute $\sum_i \dot{q}_i p_i = 2T$ and $L = T - V$ into the Hamiltonian definition:

$$H = \sum_i \dot{q}_i p_i - L.$$

Simplify:

$$H = 2T - (T - V).$$

This gives:

$$H = T + V.$$

1.11.0.6 Conclusion

The Hamiltonian H is related to the kinetic energy T and Lagrangian L by the relation:

$$H = 2T - L.$$

For time-independent systems, H represents the total energy:

$$H = T + V = E.$$