Derivation of $E = mc^2$ and Experimental Determination of Momentum of Light

Derivation of $E = mc^2$ from Shift in Center of Mass

1. System Setup

Consider a box of mass M and length L floating in free space without external forces acting on it. A photon of energy E is emitted from one side of the box and absorbed on the opposite side.

2. Momentum Conservation

When the photon is emitted, it carries momentum:

$$p_{\text{photon}} = \frac{E}{c}$$

To conserve momentum, the box recoils with equal and opposite momentum:

$$p_{\text{box}} = -Mv = -\frac{E}{c}$$

Thus, the recoil velocity v of the box is:

$$v = \frac{E}{Mc}$$

3. Displacement of the Box

The photon travels from one end of the box to the other in time t:

$$t = \frac{L}{c}$$

During this time, the box moves in the opposite direction with velocity v:

$$\Delta x = v \cdot t = \frac{E}{Mc} \cdot \frac{L}{c} = \frac{EL}{Mc^2}$$

4. Center of Mass Consideration

Initially, the center of mass (CM) of the system is at the center of the box, $\text{CM}_{\text{initial}} = 0$. After the photon is emitted, the box recoils to the left by Δx, and a hypothetical mass m equivalent to the energy E of the photon moves from one side of the box to the other.

5. Center of Mass After Emission

The positions of each mass after emission are:

- **Box (Remaining Mass $M - m$)**: Position is $-\Delta x$.
- **Hypothetical Mass m**: Position is $L - \Delta x$.

The equation for the center of mass after emission is:

$$(M - m)(-\Delta x) + m(L - \Delta x) = 0$$

Expand and simplify the equation:

$$-M\Delta x + m\Delta x + mL - m\Delta x = 0$$

$$-M\Delta x + mL = 0$$

Solve for m:

$$mL = M\Delta x$$
$$m = \frac{M\Delta x}{L}$$

6. Relating to Energy

Substitute $\Delta x = \frac{EL}{Mc^2}$ into the equation for m:

$$m = \frac{M\left(\frac{EL}{Mc^2}\right)}{L}$$

$$m = \frac{E}{c^2}$$

Thus, the energy E is equivalent to the mass m by the equation:

$$E = mc^2$$

Experimental Determination of Momentum of Light

Historical Context

Before Einstein's theory of relativity, the momentum of light was experimentally determined. The relationship between the energy and momentum of light was derived from the principles of electromagnetism and the photoelectric effect.

Momentum of Light

The momentum p of a photon is related to its energy E and frequency ν by the equation:

$$p = \frac{E}{c} = \frac{h\nu}{c}$$

where h is Planck's constant and ν is the frequency of the photon.

Experimental Evidence

The photoelectric effect demonstrated that light could impart momentum to electrons, consistent with the relationship $p = \frac{h\nu}{c}$. Experiments with radiation pressure showed that light exerts pressure on surfaces, confirming that light carries momentum.

Summary

The derivation of $E = mc^2$ using the shift in the center of mass involves conservation of momentum and the principle of the center of mass. Historical experiments on the momentum of light provided foundational evidence for understanding the energy-mass equivalence in relativity.