Differential and Total Scattering Cross Section

1. Basic Idea

When a beam of particles is sent towards a scattering center, each particle moves under a central force. Some particles go straight (large b), and some are deflected (small b).

Let:

$$b = \text{impact parameter}, \qquad \Theta = \text{scattering angle}.$$

A small range b to $b + db$ corresponds to scattering into a small range of angles Θ to $\Theta + d\Theta$.

—

2. Area relation

For the incident beam, particles that have impact parameters between b and $b + db$ fall within a ring of area

$$dA = 2\pi b\, db.$$

These particles are scattered into a cone of solid angle

$$d\Omega = 2\pi \sin\Theta\, d\Theta.$$

Since the number of particles scattered must be the same,

$$(\text{number in annulus}) = (\text{number in cone}).$$

Hence,

$$2\pi b\, db = \frac{d\sigma}{d\Omega}\,(2\pi \sin\Theta\, d\Theta).$$

—

3. Differential scattering cross section

Simplify the above equation:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|.}$$

This is the general formula for any central force.

—

4. Inverse-square (Coulomb or Gravitational) force

For force $F = -\dfrac{k}{r^2}$, we know from orbit theory:

$$\Theta = 2\arctan\!\left(\frac{k}{mbv_\infty^2}\right),$$

where

$$v_\infty = \text{velocity of particle at infinity}.$$

Now invert this to get b in terms of Θ:

$$b = \frac{k}{mv_\infty^2}\cot\!\left(\frac{\Theta}{2}\right).$$

—

5. Derive differential cross section

Differentiate b with respect to Θ:

$$\frac{db}{d\Theta} = -\frac{k}{2mv_\infty^2} \csc^2\Big(\frac{\Theta}{2}\Big).$$

Substitute in the general formula:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\Big|\frac{db}{d\Theta}\Big|.$$

After simplification:

$$\boxed{\frac{d\sigma}{d\Omega} = \Big(\frac{k}{2mv_\infty^2}\Big)^2 \frac{1}{\sin^4(\Theta/2)}.}$$

This is the **Rutherford Scattering Formula**.

—

6. Total cross section

The total scattering cross section is the total area that collects all scattered particles:

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^\pi \frac{d\sigma}{d\Omega}\, 2\pi \sin\Theta \, d\Theta.$$

For $1/r^2$ forces, $\dfrac{d\sigma}{d\Omega} \propto \dfrac{1}{\sin^4(\Theta/2)}$, so the integral diverges at small Θ (small angles). That means the total cross section is **infinite**.

In real experiments, there is a minimum measurable angle Θ_0, and we only count scattering for $\Theta > \Theta_0$.

Then,

$$\sigma(\Theta > \Theta_0) = \int_{\Theta_0}^\pi \frac{d\sigma}{d\Omega}\, 2\pi \sin\Theta \, d\Theta = \pi\Big(\frac{k}{mv_\infty^2}\Big)^2 \cot^2\Big(\frac{\Theta_0}{2}\Big).$$

—

7. Summary

Quantity	**Formula**
Deflection angle	$\Theta = 2\arctan\dfrac{k}{mbv_\infty^2}$
Differential cross section	$\dfrac{d\sigma}{d\Omega} = \Big(\dfrac{k}{2mv_\infty^2}\Big)^2 \dfrac{1}{\sin^4(\Theta/2)}$
Partial cross section ($\Theta > \Theta_0$)	$\sigma(\Theta > \Theta_0) = \pi\Big(\dfrac{k}{mv_\infty^2}\Big)^2 \cot^2\Big(\dfrac{\Theta_0}{2}\Big)$
Total cross section (ideal $1/r^2$)	Divergent (infinite)

Note: In real experiments, very small angles cannot be measured, so the total cross section is always finite in practice.

Scattering in a Central Force Field

Differential and Total Scattering Cross Sections (Classical, Central Force)

A. Kinematics and Definitions

Consider a monoenergetic beam of particles with incident speed v_∞ and number flux J (number per unit area per unit time) incident on a scattering center at the origin. Let b be the impact parameter and Θ the scattering (deflection) angle between the incoming and outgoing asymptotes. Because the force is central, the motion is planar and azimuthally symmetric.

Scattering cross section. The *differential cross section* $d\sigma/d\Omega$ is defined by

$$\frac{dN}{dt} \;=\; J\,\frac{d\sigma}{d\Omega}\,d\Omega,$$

where dN/dt is the rate of particles scattered into the solid angle element $d\Omega = \sin\Theta\, d\Theta\, d\phi$ about the direction (Θ, ϕ).

B. Mapping annulus in b to ring in solid angle

The number of incident trajectories with impact parameters between b and $b + db$ that hit the target per unit time is

$$\frac{dN}{dt} \;=\; J\,(2\pi b\, db),$$

since $2\pi b\, db$ is the area of the annulus in impact-parameter space.

Axial symmetry implies that all such trajectories scatter into a ring of polar angles between Θ and $\Theta + d\Theta$, spanning the full azimuth $0 \le \phi < 2\pi$, hence

$$d\Omega \;=\; 2\pi \sin\Theta\, d\Theta.$$

Equating the two expressions for dN/dt gives

$$J\,(2\pi b\, db) \;=\; J\,\frac{d\sigma}{d\Omega}\,(2\pi \sin\Theta\, d\Theta),$$

and therefore the *general classical formula for central forces*:

$$\boxed{\frac{d\sigma}{d\Omega} \;=\; \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|}. \tag{1}$$

This is purely kinematical; the dynamics enter through the functional relation $b \mapsto \Theta(b)$ determined by the force law.

C. Dynamics: $\Theta(b)$ from the trajectory

Energy and angular momentum conservation give

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r), \qquad L = mr^2\dot{\theta} = mv_\infty b.$$

Eliminating time,

$$\theta_0(b) = \int_{r_{\min}}^{\infty} \frac{L\,dr}{mr^2\sqrt{2(E - V(r)) - \frac{L^2}{mr^2}}}, \qquad \Theta(b) = \pi - 2\theta_0(b).$$

Once $\Theta(b)$ is known (or inverted to $b(\Theta)$), insert $b(\Theta)$ in (1) to obtain $d\sigma/d\Omega$.

D. Example: Inverse-square (Rutherford) scattering

For $F(r) = -k/r^2$ (repulsive Coulomb or attractive gravitational with suitable sign conventions), the unbound orbit is a hyperbola,

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \qquad p = \frac{L^2}{mk}, \qquad e = \sqrt{1 + \frac{2EL^2}{mk^2}} = \sqrt{1 + \frac{m^2b^2v_\infty^4}{k^2}}.$$

The asymptote satisfies $1 + e\cos\theta_\infty = 0 \Rightarrow \cos\theta_\infty = -1/e$, and the deflection angle is

$$\Theta \;=\; \pi - 2\theta_\infty \;=\; 2\sin^{-1}\!\left(\frac{1}{e}\right) \;=\; 2\arctan\!\left(\frac{k}{mbv_\infty^2}\right).$$

Inverting, one gets the impact parameter as a function of Θ:

$$\boxed{b(\Theta) = \frac{k}{mv_\infty^2}\cot\!\left(\frac{\Theta}{2}\right).} \tag{2}$$

Then

$$\frac{db}{d\Theta} = -\frac{k}{2mv_\infty^2}\csc^2\!\left(\frac{\Theta}{2}\right), \qquad \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| = \left(\frac{k}{2mv_\infty^2}\right)^2 \frac{1}{\sin^4(\Theta/2)}.$$

Hence the *Rutherford differential cross section*:

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{k}{2mv_\infty^2}\right)^2 \frac{1}{\sin^4(\Theta/2)}.} \tag{3}$$

E. Total vs. partial (integrated) cross sections

Total cross section. By definition,

$$\sigma_{\text{tot}} = \int_{4\pi} \frac{d\sigma}{d\Omega}\,d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{d\sigma}{d\Omega}\,\sin\Theta\,d\Theta.$$

For the Rutherford law (3), the integrand behaves at small angles as $\frac{d\sigma}{d\Omega} \sim \Theta^{-4}$ and $d\Omega \sim \Theta\,d\Theta\,d\phi$, so the integral diverges like $\int^{\Theta_{\min}} \Theta^{-3}\,d\Theta$. *Conclusion: the total classical cross section for a pure $1/r^2$ force is infrared divergent* (it is dominated by arbitrarily small deflection angles). Physically one introduces a cutoff (screening length, finite beam size, or minimum resolvable scattering angle).

Partial cross section above an angle cut. Define the *integrated* (or partial) cross section for deflections larger than a fixed angle $\Theta_0 > 0$:

$$\sigma(\Theta \geq \Theta_0) = \int_{\Theta_0}^{\pi}\int_0^{2\pi} \frac{d\sigma}{d\Omega}\,\sin\Theta\,d\phi\,d\Theta.$$

Using (2), there is a simple geometric identity:

$$\sigma(\Theta \geq \Theta_0) = \pi\,b(\Theta_0)^2 = \pi\Big(\frac{k}{mv_\infty^2}\cot\frac{\Theta_0}{2}\Big)^2.$$

Equivalently, inserting (3) and integrating,

$$\sigma(\Theta \geq \Theta_0) = \int_{\Theta_0}^{\pi} 2\pi\Big(\frac{k}{2mv_\infty^2}\Big)^2 \frac{\sin\Theta\,d\Theta}{\sin^4(\Theta/2)} = \pi\Big(\frac{k}{mv_\infty^2}\Big)^2 \cot^2\Big(\frac{\Theta_0}{2}\Big).$$

This finite result is what is compared with experiments when a detector has finite angular resolution or when screening suppresses very small-angle deflections.

F. Finite-range potentials

For short-range central potentials (e.g. hard sphere of radius a, or Yukawa $V(r) \propto e^{-\lambda r}/r$), the small-angle divergence is absent and the *total* cross section is finite:

$$\sigma_{\mathrm{tot}} = \int_{4\pi} \frac{d\sigma}{d\Omega}\,d\Omega < \infty.$$

As a simple example, for a hard sphere, $b \leq a$ and $\sigma_{\mathrm{tot}} = \pi a^2$.

G. Summary

- General formula (central forces): $\dfrac{d\sigma}{d\Omega} = \dfrac{b}{\sin\Theta}\Big|\dfrac{db}{d\Theta}\Big|$.
- Inverse-square (Rutherford): $\dfrac{d\sigma}{d\Omega} = \Big(\dfrac{k}{2mv_\infty^2}\Big)^2 \dfrac{1}{\sin^4(\Theta/2)}$.
- Total cross section for pure $1/r^2$: *diverges*; use an angular cutoff Θ_0 or physical screening. Then $\sigma(\Theta \geq \Theta_0) = \pi\Big(\dfrac{k}{mv_\infty^2}\Big)^2 \cot^2\big(\dfrac{\Theta_0}{2}\big)$.

1. Setup

A particle of mass m moves under a central potential $V(r)$ such that $V(r) \to 0$ as $r \to \infty$. It approaches the scattering center from infinity with speed v_∞ and impact parameter b.

2. Constants of motion

$$E = \frac{1}{2}m\big(\dot r^2 + r^2\dot\theta^2\big) + V(r), \qquad L = mr^2\dot\theta.$$

At infinity,

$$E = \frac{1}{2}mv_\infty^2, \qquad L = mv_\infty b.$$

3. Relation between r and θ

Using energy conservation,

$$\dot{r}^2 = \frac{2}{m}\big(E - V(r)\big) - \frac{L^2}{m^2 r^2}, \qquad \dot{\theta} = \frac{L}{mr^2}.$$

Hence

$$\frac{d\theta}{dr} = \frac{L/(mr^2)}{\sqrt{\frac{2}{m}\big(E - V(r)\big) - \frac{L^2}{m^2 r^2}}},$$

and the angle from infinity to the point of closest approach r_{min} is

$$\theta_0 = \int_{r_{\text{min}}}^{\infty} \frac{L\,dr}{mr^2\sqrt{2\big(E - V(r)\big) - L^2/(mr^2)}}.$$

4. Deflection (scattering) angle

Because the trajectory is symmetric,

$$\boxed{\Theta = \pi - 2\theta_0}$$

is the total scattering angle.

5. Inverse–square law $F(r) = -\dfrac{k}{r^2}$

For this potential, the unbound orbit ($e > 1$) satisfies

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \qquad p = \frac{L^2}{mk}, \qquad e = \sqrt{1 + \frac{2EL^2}{mk^2}}.$$

As $r \to \infty$, $1 + e\cos\theta_\infty = 0$ so $\cos\theta_\infty = -1/e$. Hence

$$\boxed{\Theta = \pi - 2\theta_\infty = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\arctan\left(\frac{k}{mbv_\infty^2}\right).}$$

6. Differential cross section (Rutherford formula)

The impact parameter and scattering angle are related by

$$b = \frac{k}{mv_\infty^2}\cot\left(\frac{\Theta}{2}\right).$$

Thus,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| = \left(\frac{k}{2mv_\infty^2}\right)^2 \frac{1}{\sin^4\left(\frac{\Theta}{2}\right)}.}$$

7. Summary

Quantity	**Expression**	**Meaning**
Impact parameter	$b = \dfrac{L}{mv_\infty}$	perpendicular offset
Closest distance	$r_{\min} = \dfrac{p}{1+e}$	point of closest approach
Eccentricity	$e = \sqrt{1 + \dfrac{b^2 v_\infty^4 m^2}{k^2}}$	orbit shape parameter
Deflection angle	$\Theta = 2 \arctan \dfrac{k}{mbv_\infty^2}$	total deviation
Cross section	$\dfrac{d\sigma}{d\Omega} = \left(\dfrac{k}{2mv_\infty^2}\right)^2 \dfrac{1}{\sin^4(\Theta/2)}$	Rutherford law