Teaching Notes: Thermodynamic Behavior of an Ideal Fermi Gas

Overview

We are studying a gas of **non-interacting fermions** (particles with half-integer spin like electrons, protons, etc.). The main distinguishing feature of a Fermi gas is that **no two fermions can occupy the same quantum state**, due to the **Pauli exclusion principle**. This affects how the gas behaves at both low and high temperatures.

1. What is an Ideal Fermi Gas?

- A collection of identical, non-interacting fermions.
- Each fermion has mass m, enclosed in volume V at temperature T.
- Fermions obey **Fermi-Dirac statistics**.

2. Grand Canonical Partition Function (Q)

In the grand canonical ensemble:

$$\frac{PV}{kT} = \ln Q = \sum_i \ln\left(1 + ze^{-\beta\varepsilon_i}\right)$$
$$N = \sum_i \frac{1}{z^{-1}e^{\beta\varepsilon_i} + 1}$$

where $\beta = 1/kT$ and $z = e^{\beta\mu}$.

3. Continuum Approximation (Non-Relativistic)

Using density of states in 3D:

$$\frac{P}{kT} = \frac{g}{\lambda^3} f_{5/2}(z),$$
$$\frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z)$$

where

- g = internal degrees of freedom
- $\lambda = \left(\frac{h^2}{2\pi mkT}\right)^{1/2}$
- $f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x+1} dx$

4. Equation of State and Energy

Eliminating z:

$$PV = \frac{2}{3}U$$
$$U = \frac{3}{2}NkT \cdot \frac{f_{5/2}(z)}{f_{3/2}(z)}$$

5. Specific Heat and Entropy

$$C_V = Nk\left[\frac{15}{4}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \frac{9}{4}\frac{f_{3/2}(z)}{f_{1/2}(z)}\right]$$
$$S = Nk\left[\frac{5}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z\right]$$

6. Classical Limit (High T / Low n)

When $z \ll 1$, $f_n(z) \approx z$:

$$P = \frac{NkT}{V}, \qquad U = \frac{3}{2}NkT, \qquad C_V = \frac{3}{2}Nk$$

7. Degenerate Limit ($T \to 0$)

At $T = 0$:

- Occupation number:

$$\bar{n}(\varepsilon) = \begin{cases} 1 & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$$

- Fermi energy:

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

- Internal energy:

$$U = \frac{3}{5}N\varepsilon_F$$

- Pressure:

$$P = \frac{2}{5}n\varepsilon_F$$

8. Low Temperature Expansion (Sommerfeld)

$$\mu(T) \approx \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$
$$C_V = \frac{\pi^2}{2}Nk\frac{T}{T_F}$$
$$S = \frac{\pi^2}{2}Nk\frac{T}{T_F} \to 0 \text{ as } T \to 0$$

Summary

- Fermi statistics leads to completely different low-T behavior than classical gases.
- At $T = 0$, pressure and energy are still non-zero due to the exclusion principle.
- $C_V \propto T$ at low T, consistent with experiments.