Ideal Bose Systems and Bose-Einstein Condensation

Introduction

This lecture note focuses on the behavior of an ideal gas of bosons at low temperatures, particularly the emergence of Bose-Einstein condensation (BEC). The discussion is based on Section 7.1 of Pathria's *Statistical Mechanics* (3rd edition).

1 Background: Bose-Einstein Statistics

Bosons are particles with integer spin (e.g., photons, helium-4 atoms) that obey Bose-Einstein statistics. Unlike fermions, bosons are not subject to the Pauli exclusion principle and can occupy the same quantum state.

In the grand canonical ensemble, the average number of particles in a quantum state of energy ε_i is:

$$\bar{n}_i = \frac{1}{z^{-1}e^{\beta\varepsilon_i} - 1}$$

where:

- $z = e^{\beta\mu}$ is the fugacity,
- μ is the chemical potential,
- $\beta = \frac{1}{k_B T}$.

2 Constraint on Fugacity

To ensure physical (finite) occupation numbers, the fugacity must satisfy:

$$z < 1 \quad \text{or equivalently} \quad \mu < \varepsilon_0$$

where ε_0 is the ground state energy (typically taken as zero for an ideal gas in a box).

3 Total Particle Number

The total number of particles is given by:

$$N = \sum_i \bar{n}_i = \sum_i \frac{1}{z^{-1}e^{\beta\varepsilon_i} - 1}$$

This can be split as:

$$N = N_0 + N_{\text{exc}}$$

where:

- N_0: number of particles in the ground state,
- N_{exc}: number of particles in excited states ($i > 0$).

4 Density of States Approximation

For a gas in a 3D box, the excited state sum can be approximated as an integral:

$$N_{\text{exc}} = \int_0^\infty \frac{g(\varepsilon)}{z^{-1}e^{\beta\varepsilon} - 1}\, d\varepsilon$$

where the density of states is:

$$g(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$

and the thermal de Broglie wavelength is:

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$$

Thus,

$$N_{\text{exc}} = \frac{V}{\lambda^3} g_{3/2}(z)$$

5 Critical Temperature and Condensation

The function $g_{3/2}(z)$ has a maximum value of $\zeta(3/2) \approx 2.612$ as $z \to 1$. Hence, even at $z = 1$ (maximum excited state occupation), we get:

$$N_{\text{exc}} \leq \frac{V}{\lambda^3}\zeta(3/2)$$

If the total particle number N exceeds this value, the remaining particles must go into the ground state, leading to Bose-Einstein condensation.

The critical temperature T_c is defined as the temperature where:

$$N = \frac{V}{\lambda^3}\zeta(3/2)$$

Solving for T_c:

$$T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{n}{\zeta(3/2)}\right)^{2/3} \quad \text{where } n = \frac{N}{V}$$

6 Behavior Below T_c

For $T < T_c$, the number of particles in the ground state is:

$$N_0 = N - N_{\text{exc}} = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$$

and the excited state population is:

$$N_{\text{exc}} = \frac{V}{\lambda^3}\zeta(3/2)$$

7 Summary Table

Quantity	**Expression**
Average occupation number	$\bar{n}_i = \frac{1}{z^{-1}e^{\beta\varepsilon_i} - 1}$
Excited particles	$N_{\text{exc}} = \frac{V}{\lambda^3}g_{3/2}(z)$
Ground state population ($T < T_c$)	$N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$
Critical temperature	$T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{n}{\zeta(3/2)}\right)^{2/3}$

8 Conclusion

Bose-Einstein condensation is a purely quantum phenomenon where, below a certain critical temperature, a macroscopic number of bosons occupy the same quantum state. This cannot be explained using classical physics and is a key example of how quantum statistics leads to novel collective behavior.