Thermodynamic Behavior of an Ideal Bose Gas Below T_c

Introduction

This note presents a detailed discussion of the thermodynamic properties of an ideal Bose gas below the critical temperature T_c, based on Section 7.2.B of Pathria's *Statistical Mechanics* (3rd edition).

1 Key Properties Below T_c

When $T < T_c$, Bose-Einstein condensation occurs, and a macroscopic number of bosons occupy the ground state. The fugacity $z \to 1$, implying the chemical potential $\mu \to 0$.

Total Particle Number

$$N = N_0 + N_{\text{exc}}$$

where:

- N_0: number of particles in the ground state,
- $N_{\text{exc}} = \frac{V}{\lambda^3}\zeta(3/2)$

Pressure

Since $z \to 1$, the pressure becomes:

$$P = \frac{k_B T}{\lambda^3}\zeta(5/2)$$

Note: Below T_c, pressure depends on T but not on N.

Internal Energy U

Only excited states contribute:

$$U = \frac{3}{2}\frac{V}{\lambda^3}k_B T\zeta\left(\frac{5}{2}\right) = \frac{3}{2}PV$$

Entropy S

Using $S = \frac{U + PV - \mu N}{T}$, and $\mu = 0$ below T_c:

$$S = \frac{U + PV}{T} = \frac{5}{2}\frac{V}{\lambda^3}k_B\zeta\left(\frac{5}{2}\right)$$

$$S \propto T^{3/2}$$

Specific Heat C_V

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{15}{4}\frac{V}{\lambda^3}k_B\zeta\left(\frac{5}{2}\right) \propto T^{3/2}$$

- C_V is continuous at T_c, but has a cusp (discontinuous derivative).

Summary Table

Quantity	**Expression Below T_c**
Pressure P	$\frac{k_B T}{\lambda^3}\zeta(5/2)$
Internal Energy U	$\frac{3}{2}PV$
Entropy S	$\frac{5}{2}\frac{V}{\lambda^3}k_B\zeta(5/2)$
Specific Heat C_V	$\frac{15}{4}\frac{V}{\lambda^3}k_B\zeta(5/2) \propto T^{3/2}$

Conclusion

The thermodynamic behavior of an ideal Bose gas below T_c reveals the unique features of quantum statistics, such as temperature-dependent pressure and finite energy even with increasing particle number. These are key signatures of Bose-Einstein condensation.