Thermodynamics of Radiation Fields and Ideal Quantum Gases

1. Thermodynamics of Black Body Radiation

System: Electromagnetic radiation (photon gas) in thermal equilibrium inside a cavity.

Key assumptions:

- Photons are bosons with spin 1 and zero rest mass.
- Number of photons is not conserved $\Rightarrow \mu = 0$.
- Energy levels: $\varepsilon_k = \hbar\omega_k$ where $\omega_k = c|k|$.
- Polarization: Each mode has 2 polarization states $\Rightarrow g_k = 2$.

Derivation:

- Average occupation number:

$$\bar{n}_k = \frac{1}{e^{\beta\varepsilon_k} - 1}$$

- Total internal energy:

$$U = \sum_k g_k \bar{n}_k \varepsilon_k \to \int d^3k \cdot \varepsilon_k \bar{n}_k$$

- Convert to frequency integral using:

$$g(\omega)d\omega = \frac{V\omega^2}{\pi^2 c^3} d\omega$$

Result:

$$U = aT^4 V, \quad a = \frac{8\pi^5 k_B^4}{15h^3c^3}$$

Other Thermodynamic Quantities:

- Pressure: $P = \frac{1}{3}u(T) = \frac{1}{3}\frac{U}{V}$
- Entropy: $S = \frac{4}{3}\frac{U}{T}$

2. Field of Sound Waves (Phonon Gas)

Analogous to photon gas, but with dispersion relation $\omega = c_s k$.

Assumptions:

- $\mu = 0$ (phonons not conserved)
- Three polarizations $\Rightarrow$ degeneracy factor = 3
- Debye model with linear dispersion

Derivation:

- Density of states: $g(\omega) \propto \omega^2$
- Total energy:

$$U = 9Nk_BT\left(\frac{T}{\Theta_D}\right)^3 \int_0^{\Theta_D/T} \frac{x^3}{e^x - 1}dx$$

Behavior:

- Low T: $C_V \propto T^3$ (Debye law)
- High T: $C_V \to 3Nk_B$ (Dulong-Petit law)

3. Ideal Fermi Systems

Basic Distribution:

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}$$

Fermions obey Pauli exclusion (spin-1/2).

3.3.1 Thermodynamics of Ideal Fermi Gas

At $T = 0$:

- $N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon$
- $\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$
- $U = \frac{3}{5}N\varepsilon_F$
- $P = \frac{2}{5}n\varepsilon_F$

At $T > 0$:

- Chemical potential:
$$\mu(T) \approx \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$
- Specific heat:
$$C_V \propto T$$

3.3.2 Magnetic Behavior of Ideal Fermi Gas

- Zeeman splitting: $\pm\mu B$
- Imbalance: $\Delta N \propto \mu B g(\varepsilon_F)$
- Susceptibility:
$$\chi = \mu_0 \mu^2 g(\varepsilon_F)$$

3.3.3 Electron Gas in Metals

- $n \sim 10^{28}$ electrons/m^3, $\varepsilon_F \sim 1 - 10$ eV
- $T_F \sim 10^4$ K $\Rightarrow$ Room T $\ll T_F$
- Only electrons near ε_F contribute to C_V
- $C_V^{\text{electrons}} \ll C_V^{\text{lattice}}$ at room T
- Degeneracy pressure explains stellar stability (e.g., white dwarfs)